ECE 343: Signals and Systems

Homework 3: Orthogonality Principle

In this assignment, you will be working with the vector space of functions of $t$ with $|t| < 1$. We will make this a Hilbert space by giving it the inner product

$$\langle x, y\rangle = \frac{1}{2}\int_{-1}^{1} x(t)y^*(t)dt$$

and the norm

$$||x|| = \sqrt{\langle x, x\rangle} = \sqrt{\frac{1}{2}\int_{-1}^{1} |x(t)|^2 dt}$$

Because the calculations can be tedious, you can use math software if you wish. If you do, attach a listing of the commands you used.

1. Given any two (non-zero) vectors $x$ and $y$, show that the vector $p = x - \frac{\langle x,y\rangle}{||y||^2}y$ is orthogonal to $y$. $\frac{\langle x,y\rangle}{||y||^2}y$ is called the projection of $x$ on to $y$; is the component of $x$ parallel to $y$. $p$ is called the orthogonal projection of $x$ on to $y$; it is the component of $x$ that is orthogonal to $y$.

2. The set of functions $\{\psi_0, \psi_1, \psi_2, \psi_3\} = \{1, t, t^2, t^3\}$ is in the vector space and is linearly independent. We will make an orthonormal set that spans the same subspace using the Gram-Schmidt procedure.

   (a) Starting with $\psi_0$, find $\phi_0 = \frac{\psi_0}{||\psi_0||}$. Note that $||\phi_0|| = 1$.

   (b) For the rest of the vectors, use the orthogonal projection to obtain vectors $\phi_n$ orthogonal to the others, and make them orthonormal by normalizing the them to have norm (length) 1.

   $$u_n = \psi_n - \sum_{m=0}^{n-1} \langle \psi_n, \phi_m\rangle \phi_m$$

   $$\phi_n = \frac{u_n}{||u_n||}$$

   (c) Verify that $\{\phi_0, \phi_1, \phi_2, \phi_3\}$ is an orthonormal set, i.e. show that

   $$\langle \phi_n, \phi_m\rangle = \delta[n-m] = \begin{cases} 0, & n \neq m \\ 1, & n = m \end{cases}$$

3. Approximate the function $x = \sin(\pi t)$ with the polynomials $\phi_n(t)$ from Problem 2.

   (a) Because $\{\phi_n\}$ is an orthonormal set, we can use the orthogonality principle to find weights

   $$c_n = \langle x, \phi_n\rangle.$$

   (b) Find the approximation $\hat{x}(t) = \sum_{n=0}^{3} c_n\phi_n(t)$. Find the mean squared error (*MSE*) of the approximation $||x - \hat{x}||^2$.

   (c) Find the first 4 terms of the Taylor series of $x(t)$ about $t = 0$.

   $$x_{taylor}(t) = \sum_{n=0}^{3} \frac{x^{(n)}(0)}{n!} t^n$$

   where $x^{(n)}(t)$ is the $n^{\text{th}}$derivative of $x(t)$.

   (d) Plot $x(t)$, $\hat{x}(t)$, and $x_{taylor}(t)$ on the same graph. Plot the errors $e(t) = x(t) - \hat{x}(t)$ and $e_{taylor}(t) = x(t) - x_{taylor}(t)$ on another graph. Comment about how the approximations differ and what the pros and cons of each are.